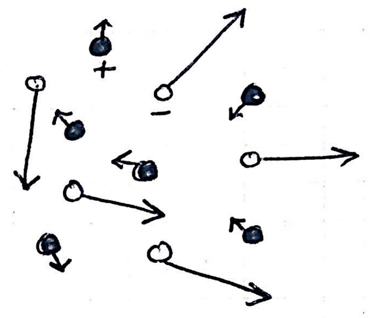
5. Plane Waves in a Tenuous Plasma

Suppose we have a "gas" of electrons and positive ions so diffuse that they do not collide with one another for times on the order of one oscillation of an incident electromagnetic wave. They then behave as "free" charges in the presence of the wave. The ions will, in general, be so massive compared to the electrons that they move negligibly compared to the motion of the electrons. The force on the electrons is given by



"tenuous plasma"

$$\vec{F} = -e(\vec{E} + \vec{v}\times\vec{B})$$

where $\vec{E}$ and $\vec{B}$ are the fields associated with the electromagnetic wave

(see page 54 of lecture notes)

Consider the relative magnitudes of the electric and magnetic forces.

$$\frac{e\vec{E}}{e\vec{v}\times\vec{B}} \sim \frac{E}{vB}$$

But for free space $E/B = \frac{1}{\mu_0}\frac{E}{H} = \frac{1}{\mu_0}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$

$$\Rightarrow \quad E/vB \sim \frac{c}{v}.$$

Thus, as long as the particles move non-relativistically (i.e. $v \ll c$), the magnetic force is negligible.

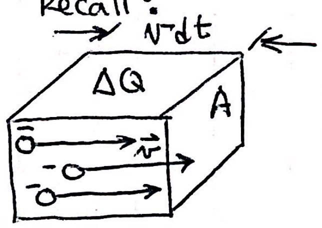
Get the velocity

$$\Rightarrow \vec{F} \approx -e\vec{E} \quad \Rightarrow m_e\frac{d\vec{v}}{dt} = -e\vec{E}$$

Since $\vec{E} = \vec{E}_0(z)e^{-i\omega t} \Rightarrow \vec{v} = -\frac{e}{m_e}\vec{E}_0\int e^{-i\omega t}\,dt$

or
$$\vec{v} = \frac{e}{i\omega m_e}\vec{E}$$

Recall:



The charge in the volume $A\,v\,dt$ is $dQ = -en_0 A v dt$ where n_0 = number density of electrons. The current is then

$$\frac{dQ}{dt} = -en_0 v A$$

and the current density is

$$J = \frac{1}{A}\frac{dQ}{dt} = -en_0 v$$

but $\vec{v}$ and $\vec{J}$ are in the same direction

Get the conductivity

Thus $\vec{J} = -en_0\vec{v}$ (see page 54) Substituting in for $\vec{v}$

$$\vec{J} = -e^2n_0/i\omega m_e\,\vec{E} = i\,n_0e^2/\omega m_e\,\vec{E}$$

$\Rightarrow$ the conductivity is then (recall $\vec{J} = \sigma \vec{E}$)

$$\boxed{\sigma = \frac{i n_o e^2}{\omega m_e}}$$

The conductivity of the plasma is imaginary!

Don't let this bother you. Since $i = e^{i\pi/2}$ this simply means that the current density $\vec{J}$ lags behind the $\vec{E}$ field by $90°$.

Now recall the equation for the complex wave number K for a wave in a conducting medium (see page 102)

$$K^2 = \mu_o \varepsilon_o \omega^2 \left(1 + \frac{i\sigma}{\varepsilon_o \omega}\right)$$

(where I have substituted $\mu \to \mu_o$ and $\varepsilon \to \varepsilon_o$ since we are in free space)

Substituting in for σ above, $K^2 = \mu_o \varepsilon_o \omega^2 \left(1 - \frac{n_o e^2}{\varepsilon_o m_e \omega^2}\right)$

or $\boxed{K^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)}$ where $c^2 = \frac{1}{\mu_o \varepsilon_o}$ and $\boxed{\omega_p^2 \equiv \frac{n_o e^2}{\varepsilon_o m_e}}$

ω_p is called the plasma frequency

Case #1: propogation above the plasma frequency $\omega > \omega_p$

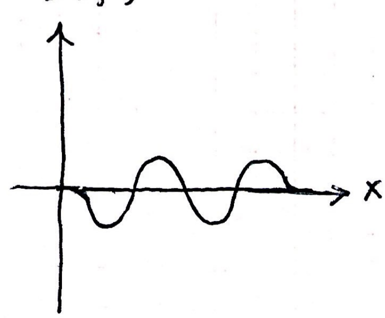
$\Rightarrow K = \frac{\omega}{c}\left(1 - \omega_p^2/\omega^2\right)^{1/2}$ which is real. Therefore, the wave propogates unattenuated, $\vec{E} = \vec{E}_o e^{-i(\omega t - Kz)}$, with velocity $v = \frac{\omega}{K} = \frac{c}{(1-\omega_p^2/\omega^2)^{1/2}} > c$ (Surprised?)

or the index of refraction is $n \equiv c/v = (1 - \omega_p^2/\omega^2)^{1/2} < 1$

This is not a problem since special relativity only tells us that information can't be propogated faster than c and a plane, monochromatic wave doesn't carry any information; it's very boring. Once you know its amplitude and phase you can predict what it's going to do forever and ever.

Suppose we want to send a message in the form of pulse of radiation (see figure at right). We can express this pulse as a sum (integral) of monochromatic waves of different amplitudes and frequencies, ie

$$E(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\omega(k)t}\, dk$$



where we have allowed for the possibility that k can be a function of ω (or equivalently ω a function of k)

The amplitudes $A(k)$ are related to the initial wave form by a Fourier transform

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(x,0)\, e^{-ikx}\, dx$$

(see Mechanics notes page 35 for a similar result for a Fourier series)

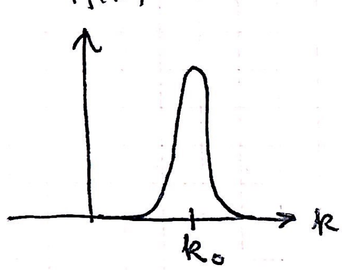
It can be shown that in order to make a wave pulse of finite extent Δx (where stands for the rms width about the mean center of the pulse), one requires waves with a spread in wavenumber Δk (rms width of $A(k)$ about the mean wavenumber) such that

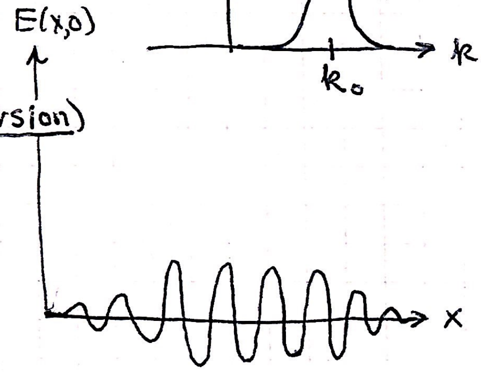
$$\Delta x \Delta k \geq \frac{1}{2} \quad \text{or} \quad \Delta k \geq \frac{1}{2\Delta x}$$

A similar relation holds for the time duration of a pulse Δt and the frequency spread

$$\Delta t \Delta \omega \geq \frac{1}{2} \Rightarrow \Delta\omega \geq \frac{1}{2\Delta t}$$

(Note: in quantum mechanics the momentum of a particle is $p = \hbar k$ and the energy is $E = \hbar\omega$. Direct substitution of these into the above inequalities gives the Heisenberg uncertainty relations, ie. $\Delta p \Delta x \geq \hbar/2$ and $\Delta E \Delta t \geq \hbar/2$)





Now suppose we construct a rather broad pulse with a small Δk, ie. $A(k)$ is a highly peaked function (see figure at right). If the frequency is a function of k then the velocity of the monochromatic waves $v = \omega/k$ is different for different waves. One result of this is that the pulse of radiation will spread out in time (dispersion). If the pulse is initially broad, as in the figure, this spreading won't occur too rapidly. Another effect is that the pulse as a whole travels at a rather different velocity as the velocity of its individual components. We now derive this velocity. Since $A(k)$ is peaked about k_0 let's expand $\omega(k)$ about k_0

$$\omega(k) \cong \omega_0 + \left.\frac{d\omega}{dk}\right|_0 (k - k_0)$$

Then the expression for $E(x,t)$ on the bottom of the previous page becomes

$$E(x,t) \cong e^{-i\left(\omega_0 - \left.\frac{d\omega}{dk}\right|_0 k_0\right)t} \int_{-\infty}^{\infty} A(k)\, e^{ikx - i\left.\frac{d\omega}{dk}\right|_0 kt}$$

or $E(x,t) \propto f\left(x - \left.\frac{d\omega}{dk}\right)_0 t\right)$

$\Rightarrow$ The waveform $E(x,t)$ travels with the group velocity

$$\boxed{v_g = \frac{d\omega}{dk}}$$

The velocity of a monochromatic wave which we discussed previously is called the phase velocity

$$\boxed{v_p = \frac{\omega}{k}}$$

question: can you show that $v_g = \dfrac{c}{[n(\omega) + \omega(dn/d\omega)]}$?

where $n(\omega)$ is the index of refraction.

For a tenuous plasma we found (see page 107) that

$$K^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} \Rightarrow \omega = \sqrt{\omega_p^2 + K^2c^2}$$

thus $$\frac{d\omega}{dK} = \frac{c^2 K}{\sqrt{\omega_p^2 + K^2c^2}} = \frac{c}{\sqrt{1 + \frac{\omega_p^2}{K^2c^2}}} = \frac{c}{\sqrt{1 + \frac{\omega_p^2}{(\omega^2 - \omega_p^2)}}} < c$$

(so long as $\omega > \omega_p$ as we assumed for case#1) Good!

For a wave polarized in the $\hat{x}$ direction $\vec{E}_0 = E_{0x}\hat{x}$

Then $\vec{\nabla} \times \vec{E} = -\mu_0 \frac{d\vec{H}}{dt} \Rightarrow iKE_{0x}\,\hat{z} \times \hat{x} = i\omega\mu_0 \vec{H}_0$ (see page 97)

$$\Rightarrow \vec{H}_0 = \frac{K}{\mu_0\omega} E_{0x}\,\hat{y} \Rightarrow E_{0x}/H_{0y} = \frac{\mu_0\omega}{K} = \mu_0 c \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}$$

$$\Rightarrow E_{0x}/H_{0y} = \sqrt{\frac{\mu_0}{\epsilon_0}}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}$$

Case #2: Propogation of a wave below the plasma freqnency ($\omega < \omega_p$)

$$\Rightarrow K^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) = -\frac{1}{c^2}(\omega_p^2 - \omega^2)$$

$\Rightarrow K = i/c \sqrt{\omega_p^2 - \omega^2}$ which is purely imaginary

Thus electric (and magnetic) fields fall off exponentially in the plasma, ie. $\vec{E} = \vec{E}_0 e^{-z/\delta_{plasma}} e^{-i\omega t}$ where $\delta_{plasma} = \dfrac{c}{\sqrt{\omega_p^2 - \omega^2}}$

Therefore, the wave doesn't propogate at all.

For low frequencies, $\omega \ll \omega_p$, $\delta_{plasma} \approx c/\omega_p$

In the laboratory, plasma densities are in the range 10^{12} to $10^{16}/cm^3$.

$$\Rightarrow \omega_p = 6\times10^{10} \text{ to } 6\times10^{12}\,\text{sec}^{-1} \Rightarrow \delta_{plasma} = .5\,\text{cm to } 5\times10^{-3}\,\text{cm}$$

at low frequencies. Thus low frequency fields, in particular static fields, are expulsed from plasmas.

<u>Question</u>: explain, physically, why this is so.

<u>Question</u>: If the amplitude decays exponentially, where does the energy go? Joule heating? No!

$$P_{joule} = \vec{J}\cdot\vec{E} = \sigma\vec{E}\cdot\vec{E} \qquad \text{now } \sigma\vec{E} \propto e^{-i\omega t + i\pi/2}$$

and $\vec{E} \propto e^{-i\omega t}$ $\Rightarrow$ $\mathrm{Re}(\sigma\vec{E}) \propto \cos(\omega t - \pi/2) = \sin\omega t$ and

$\mathrm{Re}(\vec{E}) \propto \cos\omega t$ $\Rightarrow \langle \vec{J}\cdot\vec{E} \rangle \propto \langle \sin\omega t \cos\omega t \rangle = 0$.

That is, since $\vec{J}$ and $\vec{E}$ are always 90° out of phase, no energy is dissipated. This makes sense. Remember we initially assumed there were no collisions to soak up energy.

So what happened to the energy? It was <u>reflected</u> (we'll deal with reflections next week). A wave incident on the plasma with $\omega < \omega_p$ is reflected back out.

<u>Question</u>: Show that no power is transmitted to the plasma by evaluating the average value of the Poynting vector.

<u>Question</u>: In the ionosphere $n_0 \sim 10^{11}/m^3$. Are there many radio astronomers who work at frequencies below 1 megaherz?